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On patching map

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1 Main result

Let A and B be two topological spaces. We consider the disjoint union $X := A \sqcup B$, and assume the following:

Setting 1. *The topology on X satisfies the following conditions (a) and (b).*

(a) The induced topology from X to A coinside with the original topology on A.
(b) The induced topology from X to B coinside with the original topology on B.

Note that such a topology on X is not unique. Because, to specify the topology on X, we need to specify how A and B connect each other. Roughly speaking,

$$(\text{topology on } X) = (\text{topology on } A) + (\text{topology on } B) + (\text{how to connect}).$$

The motivation to introduce a patching map is to describe the information of (how to connect). Let $\mathcal{O}_A$ (resp. $\mathcal{O}_B$) denote the set of all open subsets in A (resp. B). Then, $\mathcal{O}_A$ (resp. $\mathcal{O}_B$) describes the topology on A (resp. B). On the other hand, the information of connection is described by such a following map.

Defintion 2. *A* **patching map** *is a map $\mu : \mathcal{O}_A \to \mathcal{O}_B$ having the following properties:*

(P1) $\mu(A) = B$,
(P2) $\mu(U \cap V) = \mu(U) \cap \mu(V) \qquad (U, V \in \mathcal{O}_A)$,

Proposition 3. *A patching map μ satisfies the following.*

$$U \subset V \;\Rightarrow\; \mu(U) \subset \mu(V).$$

Proof.

$$\begin{aligned} U \subset V &\iff U \cap V = U \\ &\implies \mu(U \cap V) = \mu(U) \\ &\iff \mu(U) \cap \mu(V) = \mu(U) \\ &\iff \mu(U) \subset \mu(V). \end{aligned}$$

□

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Defintion 4. *Let $\mu_{B,A} : \mathcal{O}_A \to \mathcal{O}_B$ and $\mu_{A,B} : \mathcal{O}_B \to \mathcal{O}_A$ be two patching maps. The pair $(\mu_{A,B}, \mu_{B,A})$ is said to be* **compatible** *if the following two conditions are satisfied.*

(1) $U \subset \mu_{B,A}(\mu_{A,B}(U))$ *(for $U \in \mathcal{O}_B$),*
(2) $U \subset \mu_{A,B}(\mu_{B,A}(U))$ *(for $U \in \mathcal{O}_A$).*

Let $\mu := (\mu_{A,B}, \mu_{B,A})$ be a pair of patching maps. We regard μ as a map $\mu : \mathcal{O}_A \sqcup \mathcal{O}_B \to \mathcal{O}_A \sqcup \mathcal{O}_B$ defined by

$$\mu(U) := \begin{cases} \mu_{B,A}(U) & (U \in \mathcal{O}_A) \\ \mu_{A,B}(U) & (U \in \mathcal{O}_B) \end{cases}$$

Then the compatibility condition is equivalent to

$$U \subset \mu^2(U) \qquad (\text{for } U \in \mathcal{O}_A \sqcup \mathcal{O}_B).$$

Remark 5. *Note that we regard that $\emptyset \in \mathcal{O}_A$ and $\emptyset \in \mathcal{O}_B$ are distinct.*

The next is our main result.

Theorem 6. *There is the following correspondence. (See Definitions 12, 14).*

$$\mathrm{Top}(A, B) \overset{1:1}{\longleftrightarrow} \mathrm{PM}(A, B)$$

Here, we define

$$\mathrm{Top}(A, B) := \{\text{topology on } X \text{ satisfying Setting 1}\},$$
$$\mathrm{PM}(A, B) := \{\text{compatible pair of patching maps}\}.$$

2 Trivial patching map

In many cases, we consider the case that A or B is open in X. In this case, it is enough to consider 'one way patching map'. Roughly speaking, if A is open, the information of connection is described by only a patching map from $\mathcal{O}_A$ to $\mathcal{O}_B$.

Defintion 7. *A patching map $\mu : \mathcal{O}_A \to \mathcal{O}_B$ is said to be* **trivial** *if*

$$\mu(U) = B \qquad (\text{for any } U \in \mathcal{O}_A).$$

Then we have

Proposition 8. *Assume X has a topology satisfying Setting 1, and $(\mu_{A,B}, \mu_{B,A})$ is the corresponding pair of patching map. Then,*

$$A \text{ is open in } X \iff \mu_{A,B} \text{ is trivial}$$

Proposition 9. *If $\mu_{A,B}$ is trivial, any pair $(\mu_{A,B}, \mu_{B,A})$ is compatible.*

From Theorem 6 and Propositions 8 and 9, we obtain

Theorem 10. *There is a one to one correspondings.*

$$\mathrm{Top}(A,B;\ A \textit{ is open}) \xleftrightarrow{1:1} \mathrm{PM}(A \to B).$$

Here, we define

$$\begin{aligned}\mathrm{Top}(A,B;\ A \text{ is open}) &:= \{\mathcal{O} \in \mathrm{Top}(A,B) \mid A \text{ is open in } X\}, \\ \mathrm{PM}(A \to B) &:= \{\text{patching map } \mu : \mathcal{O}_A \to \mathcal{O}_B\}.\end{aligned}$$

3 Interior operator of induced topology

In this section, we prepare lemma for later purpose.

Let X be a topological space. We denote by Int_X (resp. Cl_X) the interior (resp. closure) operator of X. Take any subset $A \subset X$ with induced toplogy, and denote by Int_A (resp. Cl_A) the interior (resp. closure) operator of A. We want to describe Int_A (resp. Cl_A) by using Int_X (resp. Cl_X).

Lemma 11. *Let $B = X \setminus A$. For a subset $U \subset A$, we have:*

(1) $\mathrm{Cl}_A(U) = A \cap \mathrm{Cl}_X(U)$,
(2) $\mathrm{Int}_A(U) = A \cap \mathrm{Int}_X(B \sqcup U)$.

Proof. (1) The inclusion '$\subset$' comes from:

$$U \subset A \cap \mathrm{Cl}_X(U),$$

because $A \cap \mathrm{Cl}_X(U)$ is closed in A. On the other hand, to see the inclusion '$\supset$', we take a closed subset F in X such that $\mathrm{Cl}_A(U) = A \cap F$. Since F is closed in X, the inclusion

$$U \subset \mathrm{Cl}_A(U) \subset F$$

implies $\mathrm{Cl}_X(U) \subset F$. Thus we obtain

$$A \cap \mathrm{Cl}_X(U) \subset A \cap F = \mathrm{Cl}_A(U).$$

This is what we wanted.

(2) This follows from:

$$\begin{aligned}\mathrm{Int}_A(U) &= A \setminus \mathrm{Cl}_A(A \setminus U) \\ &= A \setminus \mathrm{Cl}_X(A \setminus U) \\ &= A \cap \mathrm{Int}_X(X \setminus (A \setminus U)) \\ &= A \cap \mathrm{Int}_X(B \sqcup U).\end{aligned}$$

$\square$

4 Correspondence

Defintion 12. *We define a map $\Psi : \mathrm{Top}(A, B) \to \mathrm{PM}(A, B)$ as follows:*
For a topology in $\mathrm{Top}(A, B)$, we denote by Int_X the interior operator. We define patching maps by

$$\mu_{A,B}(U) := A \cap \mathrm{Int}_X(A \sqcup U) \qquad (\text{for } U \in \mathcal{O}_B),$$
$$\mu_{B,A}(U) := B \cap \mathrm{Int}_X(B \sqcup U) \qquad (\text{for } U \in \mathcal{O}_A).$$

Proposition 13. *The above map Ψ is well-defined. In other words, we have the following:*

(1) $\mu_{A,B}(U) \in \mathcal{O}_A, \quad \mu_{B,A}(U) \in \mathcal{O}_B,$
(2) *$\mu_{A,B}$ and $\mu_{B,A}$ satisfy the axiom of patching map.*
(3) *$(\mu_{A,B}, \mu_{B,A})$ are compatible.*

Proof. (1) This is clear by definition.
(2) We need to verify the following four conditions.

(a) $\mu_{A,B}(B) = A$ (b) $\mu_{B,A}(A) = B$
(c) $\mu_{A,B}(U \cap V) = \mu_{A,B}(U) \cap \mu_{A,B}(V)$ (d) $\mu_{B,A}(U \cap V) = \mu_{B,A}(U) \cap \mu_{B,A}(V)$

Here, (a) and (b) are clear. The condition (c) and (d) comes from:

$$\mathrm{Int}_X(U \cap V) = \mathrm{Int}_X(U) \cap \mathrm{Int}_X(V).$$

(3) For an open set $U \in \mathcal{O}_A$, we have:

$$\begin{aligned}
\mu^2(U) &= A \cap \mathrm{Int}_X(A \sqcup (B \cap \mathrm{Int}_X(B \sqcup U))) \\
&= A \cap \mathrm{Int}_X(A \cup \mathrm{Int}_X(B \sqcup U)) \\
&\supset A \cap \mathrm{Int}_X(\mathrm{Int}_X(B \sqcup U)) \\
&= \mathrm{Int}_A(U) \\
&= U.
\end{aligned}$$

Similary, we also obtain $\mu^2(U) \supset U$ for $U \in \mathcal{O}_B$. Thus the pair is compatible.

□

Defintion 14. *We define a map $\Phi : \mathrm{PM}(A, B) \to \mathrm{Top}(A, B)$ as follows:*
Take any $\mu = (\mu_{A,B}, \mu_{B,A}) \in \mathrm{PM}(A, B)$. For a subset $U \subset X$, we define:

$$U \text{ is open in } X \overset{def}{\iff} U \text{ satisfies the following four conditions.}$$

(a) $U_A \in \mathcal{O}_A$, (b) $U_B \in \mathcal{O}_B$,
(c) $U_A \subset \mu(U_B)$, (d) $U_B \subset \mu(U_A)$.

Here, we put $U_A := A \cap U$ and $U_B := B \cap U$.

Proposition 15. *The above map Φ is well-defined. In other words, we have the followings:*

(1) *The family of above open sets satisfies the axiom of open subsets.*

(2) *The topology satisfies Setting 1.*

Proof. (1) We put

$$\begin{aligned} \mathcal{O}_X &:= \{U \subset X \mid \ U \text{ satisfies (a),(b),(c) and (d)}\} \\ \mathcal{O}_a &:= \{U \subset X \mid \ U \text{ satisfies (a)}\} \\ \mathcal{O}_b &:= \{U \subset X \mid \ U \text{ satisfies (b)}\} \\ \mathcal{O}_c &:= \{U \subset X \mid \ U \text{ satisfies (c)}\} \\ \mathcal{O}_d &:= \{U \subset X \mid \ U \text{ satisfies (d)}\}. \end{aligned}$$

Then, we have $\mathcal{O}_X = \mathcal{O}_a \cap \mathcal{O}_b \cap \mathcal{O}_c \cap \mathcal{O}_d$. Thus, to show $\mathcal{O}_X$ satisfies the axiom of open subsets, it is sufficient to show that $\mathcal{O}_a$, $\mathcal{O}_b$, $\mathcal{O}_c$ and $\mathcal{O}_d$ satisfy the axiom of open subsets. Here, $\mathcal{O}_a$ and $\mathcal{O}_b$ clearly satisfy the axiom of open subsets. Let us see that $\mathcal{O}_c$ satisfies the axiom.

(i) $\emptyset \in \mathcal{O}_c$.
This comes from

$$\emptyset \subset \mu(\emptyset).$$

(ii) $U, V \in \mathcal{O}_c \implies U \cap V \in \mathcal{O}_c$.
The condition $U, V \in \mathcal{O}_c$ implies $U_A \subset \mu(U_B)$ and $V_A \subset \mu(V_B)$. So the condition $U \cap V \in \mathcal{O}_c$ comes from:

$$(U \cap V) \cap A \quad = \quad U_A \cap V_A \quad \subset \quad \mu(U_B) \cap \mu(V_B) \quad = \quad \mu((U \cap V) \cap B).$$

(iii) $\bigcup_\lambda U_\lambda \in A$ for any family $\{U_\lambda\} \subset \mathcal{O}_c$.
Take any family $\{U_\lambda\} \subset \mathcal{O}_c$, and put $U := \bigcup_\lambda U_\lambda$. Since we have $\mu(U_\lambda \cap B) \subset \mu(U \cap B)$ by Proposition 3, the condition $U \in \mathcal{O}_c$ comes from:

$$U \cap A \quad = \quad \bigcup_\lambda (U_\lambda \cap A) \quad \subset \quad \bigcup_\lambda \mu(U_\lambda \cap B) \quad \subset \quad \bigcup \mu(U \cap B) \quad = \quad \mu(U \cap B).$$

Thus, we have proved that $\mathcal{O}_c$ satisfies the axiom. Similary we can prove that $\mathcal{O}_d$ satisfies the axiom.

(2) Let $\mathcal{O}_{X\downarrow A}$ denote the family of open sets in the sense of induced topology on A from X. We want to see that $\mathcal{O}_{X\downarrow A} = \mathcal{O}_A$. This equation comes from:

$$U \in \mathcal{O}_{X\downarrow A} \implies {}^{\exists}V \subset B \text{ such that } U \sqcup V \in \mathcal{O}_X \implies U \in \mathcal{O}_A,$$

and

$$U \in \mathcal{O}_A \implies U \sqcup \mu(U) \in \mathcal{O}_X \implies U \in \mathcal{O}_{X\downarrow A}.$$

□

5 Basic Lemma

Let $\mu = (\mu_{A,B}, \mu_{B,A})$ be a compatible pair of patching map. And suppose that X has the topology defined as in Definition 14. We denote by Int_X the interior operator of X.

Lemma 16. *For a subset $U \subset X$, we have the following.*

(i) $A \cap \mathrm{Int}_X(U) = U_A^i \cap \mu(U_B^i)$,
(ii) $B \cap \mathrm{Int}_X(U) = U_B^i \cap \mu(U_A^i)$.

Here we define

$$U_A^i := \mathrm{Int}_A(U \cap A),$$
$$U_B^i := \mathrm{Int}_B(U \cap B).$$

Proof. We put

$$V_A := U_A^i \cap \mu(U_B^i),$$
$$V_B := U_B^i \cap \mu(U_A^i),$$
$$V := V_A \sqcup V_B.$$

Then, the conditions (i) and (ii) in Lemma 16 is equivalent to

$$\mathrm{Int}_X(U) = V.$$

So, it is enough to show the following three conditions.

(1) $V \in \mathcal{O}_X$,
(2) $V \subset U$,
(3) $U' \in \mathcal{O}_X$ and $U' \subset U \Longrightarrow U' \subset V$.

(Step 1) We show $V \in \mathcal{O}_X$.
To show this, it is enough to show the following four conditions.

(a) $V_A \in \mathcal{O}_A$, (b) $V_B \in \mathcal{O}_B$,
(c) $V_A \subset \mu(V_B)$, (d) $V_B \subset \mu(V_A)$.

Here, the conditions (a) and (b) are clear. The condition (c) follows from:

$$V_A = U_A^i \cap \mu(U_B^i) \subset \mu^2(U_A^i) \cap \mu(U_B^i) = \mu(\mu(U_A^i) \cap U_B^i) = \mu(V_B).$$

By exchanging A and B, we also obtain the conditions (d).

(Step 2) We show $V \subset U$.
The inclusion $V_A \subset U$ comes from:

$$V_A \subset U_A^i \subset U.$$

Similary, we also obtain $V_B \subset U$. Thus, we obtain $V \subset U$.

(Step 3) Finally let us show that $U' \in \mathcal{O}_X$ and $U' \subset U \Longrightarrow U' \subset V$.
We put $U'_A = U' \cap A$, and show $U'_A \subset V_A$. Then, by exchanging A and B, we will also have $U'_B \subset V_B$, and so $U'_A \sqcup U'_B \subset V_A \sqcup V_B$. This is what we want.

Since we have $V_A = U_A^i \cap \mu(U_B^i)$, to show $U'_A \subset V_A$, it is enough to see:

$$\text{(a) } U'_A \subset U_A^i, \qquad \text{(b) } U'_A \subset \mu(U_B^i).$$

Since $U'_A \in \mathcal{O}_A$, the inclusion $U'_A \subset U_A$ implies

$$U'_A \subset U^i_A.$$

So, the condition (a) is proved. By exchanging A and B, we also obtain $U'_B \subset U^i_B$. On the other hand, $U' \in \mathcal{O}_X$ implies $U'_A \subset \mu(U'_B)$. Thus, we have

$$U'_A \subset \mu(U'_B) \subset \mu(U^i_B).$$

The condition (b) is proved. And so, Step 3 is completed. □

6 Inverse

In the previous section, we defined maps $\Psi : \mathrm{Top}(A, B) \to \mathrm{PM}(A, B)$ and $\Phi : \mathrm{PM}(A, B) \to \mathrm{Top}(A, B)$. Now, let us see they are inverse maps each other.

Lemma 17. *The following composition map is identity.*

$$\mathrm{PM}(A, B) \xrightarrow{\Phi} \mathrm{Top}(A, B) \xrightarrow{\Psi} \mathrm{PM}(A, B).$$

Proof. Take any $\mu \in \mathrm{PM}(A, B)$, and put $\mu^\# := \Psi(\Phi(\mu))$. We want to show $\mu^\# = \mu$. For an open subset $W \in \mathcal{O}_A$, we have:

$$\mu^\#(W) = \mu^\#_{B,A}(W) = B \cap \mathrm{Int}_X(B \sqcup W). \tag{1}$$

Here, Int_X is the interior operator with respect to the topology $\Phi(\mu)$. We put

$$U := B \sqcup W.$$

Then, Lemma 16 implies

$$B \cap \mathrm{Int}_X(U) = U^i_B \cap \mu(U^i_A). \tag{2}$$

Moreover, we have

$$U^i_A = \mathrm{Int}_A(A \cap U) = \mathrm{Int}_A(A \cap (B \sqcup W)) = W,$$
$$U^i_B = \mathrm{Int}_B(B \cap U) = \mathrm{Int}_B(B \cap (B \sqcup W)) = B.$$

From (1) and (2), we obtain:

$$\mu^\#(W) = B \cap \mu(W) = \mu(W).$$

So, we proved

$$\mu^\#|_{\mathcal{O}_A} = \mu|_{\mathcal{O}_A}.$$

By exchanging A and B, we also obtain

$$\mu^\#|_{\mathcal{O}_B} = \mu|_{\mathcal{O}_B}.$$

Thus $\mu^\# = \mu$ is proved. □

Lemma 18. *The following composition map is identity.*

$$\mathrm{Top}(A, B) \xrightarrow{\Psi} \mathrm{PM}(A, B) \xrightarrow{\Phi} \mathrm{Top}(A, B).$$

To show Lemma 18, we need the following:

Lemma 19. *Assume X has a topology satisfying Setting 1. We denote by* Int_X*, the interior operator of* X*. For any subset $U \subset X$, we have:*

$$\mathrm{Int}_X(A \cup \mathrm{Int}_X(A \cup U)) = \mathrm{Int}_X(A \cup U).$$

Proof. The inclusion '$\subset$' comes from

$$A \cup \mathrm{Int}_X(A \cup U) \subset A \cup U.$$

On the other hand, the inclusion '$\supset$' comes from

$$A \cup \mathrm{Int}_X(A \cup U) \supset \mathrm{Int}_X(A \cup U).$$

□

Finally, let us show Lemma 18.

Proof. Take any $\mathcal{O}_X \in \mathrm{Top}(A, B)$, and put $\mathcal{O}_X^\# := \Phi(\Psi(\mathcal{O}_X))$. We denote by Int_X (resp. $\mathrm{Int}_X^\#$) the interior operator with respect to $\mathcal{O}_X$ (resp. $\mathcal{O}_X^\#$). It is enough to show $\mathrm{Int}_X^\# = \mathrm{Int}_X$. For a subset $U \subset X$, we shall see

$$A \cap \mathrm{Int}_X^\#(U) = A \cap \mathrm{Int}_X(U). \tag{3}$$

Then, by exchanging A and B, we can also obtain $B \cap \mathrm{Int}_X^\#(U) = B \cap \mathrm{Int}_X(U)$, so $\mathrm{Int}_X^\# = \mathrm{Int}_X$ will follow.

Let us see (3). By Lemma 16, we have

$$A \cap \mathrm{Int}_X^\#(U) = U_A^i \cap \mu(U_B^i).$$

Here we need the following claims.

Claim 1. $U_A^i = A \cap \mathrm{Int}_X(B \cup U)$.
Claim 2. $\mu(U_B^i) = A \cap \mathrm{Int}_X(A \cup U)$.

Once, these claims are proved, we have

$$\begin{aligned} A \cap \mathrm{Int}_X^\#(U) &= U_A^i \cap \mu(U_B^i) \\ &= A \cap \mathrm{Int}_X(B \cup U) \cap \mathrm{Int}_X(A \cup U) \\ &= A \cap \mathrm{Int}_X((B \cup U) \cap (A \cup U)) \\ &= A \cap \mathrm{Int}_X(U). \end{aligned}$$

This is what we wanted.

Let us show Claim 1. This comes from:

$$\begin{aligned} U_A^i &= \mathrm{Int}_A(A \cap U) \\ &= A \cap \mathrm{Int}_X(B \sqcup (A \cap U)) && \text{(Lemma 11(2))} \\ &= A \cap \mathrm{Int}_X(B \cup U). \end{aligned}$$

By exchanging A and B, we also obtain

$$U_B^i = B \cap \mathrm{Int}_X(A \cup U).$$

Then, Claim 2 comes from:

$$\begin{aligned}
\mu(U_B^i) &= A \cap \mathrm{Int}_X(A \sqcup U_B^i) \\
&= A \cap \mathrm{Int}_X(A \sqcup (B \cap \mathrm{Int}_X(A \cup U))) \\
&= A \cap \mathrm{Int}_X(A \cup \mathrm{Int}_X(A \cup U)) \\
&= A \cap \mathrm{Int}_X(A \cup U). && \text{(Lemma 19)}
\end{aligned}$$

Thus, Lemma is proved. $\square$